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LETTER TO THE EDITOR

Screening approximation and the Kolmogorov spectrum of homogeneous isotropic turbulence

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Abstract. If the physical picture inherent in Kolmogorov's description of turbulence is given a mathematical implementation, then a finite theory of turbulence yielding the Kolmogorov spectrum can be constructed. We discuss how this can be done in a screening approximation which exploits the viscoelastic effect.

Homogeneous isotropic turbulence seems to support the Kolmogorov scaling (Kolmogorov 1941) to a very high degree of accuracy for the low-order velocity structure factors. In particular for the two-point correlation $C(k)$ which is related to the energy spectrum $E(k)$, Kolmogorov scaling asserts that

$$k^2C(k) = E(k) = K\epsilon^{2/3}k^{-5/3} \tag{1}$$

where K is a universal constant and ϵ is the rate at which energy is fed to maintain the turbulence. The validity of (1) is supposed to be in the inertial range characterized by momenta which are much larger than the momenta at which energy is injected (large spatial scales) and much smaller than the momenta at which energy is dissipated by molecular viscosity (small spatial scales). The dimensional analysis which leads to (1) also provides the momentum dependence of an effective relaxation rate $\Gamma(k)$ for the turbulent fluctuations as

$$\Gamma(k) = \Gamma\epsilon^{1/3}k^{2/3} \tag{2}$$

where Γ is a universal constant.

One of the important problems in the analytic theory of turbulence has been the question of obtaining (1) and (2) from the Navier–Stokes equation for incompressible fluids. As stated very clearly by Leslie (Leslie 1972), one should start out with the scaling ansatz for $C(k)$ and $\Gamma(k)$, namely

$$C(k) = C_0k^{-m} \tag{3a}$$

$$\Gamma(k) = \Gamma_0k^{n} \tag{3b}$$

and then establish that $n = 1/3$ and $m = 11/3$ as required by (1) and (2). The difficulty in doing this analysis was first noted by Edwards who showed that the perturbative treatment of velocity responses and correlation functions in the Navier–Stokes equation led to divergent integrals (Leslie 1972). Overcoming this problem with divergent integrals has been the main challenge.

Renormalization group (Yakhot and Orszag 1986) provided a way out of the difficulty. The starting point was a randomly stirred Navier–Stokes equation (Forster *et al* 1976, De Dominicis and Martin 1979) and the renormalization group recursion relation was obtained by eliminating wave numbers in a band in the high momentum range. The divergent which comes from the small wave numbers (infrared range) is then avoided. A mode coupling calculation which exploits the form of the random force in evaluating the response integrals yields identical answers (Bhattacharjee 1988). A common criticism of all such calculations is the special choice of the random force that one has to make in order to obtain the Kolmogorov spectrum.

In this note, we point out that if the mathematics follows the physical picture of Kolmogorov in every detail, then one can get the Kolmogorov spectrum from the Navier–Stokes equation without having to worry about the exact nature of the random force. The necessity of the random external force is to maintain the turbulence and provide the distribution over which averaging has to be done to calculate the response and correlation functions. The physical picture of Kolmogorov requires:

(i) lossless transfer of energy across various momentum scales in the inertial range

(ii) the largest scales (energy containing eddies) to simplify the small scales (energy dissipating eddies) and not to affect the dynamics at the small scales.

It is well known (Orszag 1973) that in the Eulerian picture, the difficulty of ensuring the second constraint above leads to the infrared divergence. The way out could be a Lagrangian picture (Kraichnan 1966) or a semi-Lagrangian one (Horner and Lipowsky 1979), however, calculations in the Lagrangian picture are tremendously difficult. Our point is that by using the screening produced by viscoelastic effects (Crow 1968), it is possible to eliminate the infrared divergence and implement the scheme proposed by Leslie.

We consider the forced Navier–Stokes equation

$$\dot{V}_i(k) = -P_{ijn}(k)v_j(p)v_n(k-p) + \nu k^2 v_i(k) + f_i \tag{4a}$$

with

$$P_{ijn}(k) = \tfrac{1}{2}[k_j P_{in}(k) + k_n P_{ij}(k)] \tag{4b}$$

where $P_{ij}(k)$ is the projection operator $\delta_{ij} - k_i k_j/k^2$. The principal elements of the calculation involve the response function $F(k,\omega)$ and the correlation function $C(k,\omega) = \langle v_i(k,\omega)v_i(-k,-\omega)\rangle$, where the angular brackets denote averaging over the statistics of the random force. The response function $F(k,\omega)$ is to the zeroth order given by $G_0(k,\omega) = (-\mathrm{i}\omega + \nu k^2)^{-1}$. The full response function is given by Dysons' equation

$$\begin{aligned} G(k,\omega) &= -\mathrm{i}\omega + \nu_0 k^2 + \textstyle\sum(k,\omega) \\ &\simeq -\mathrm{i}\omega + \textstyle\sum(k,\omega) \end{aligned} \tag{5}$$

where the last line holds in the inertial range, where the molecular viscosity is negligible compared to the eddy viscosity $\sum(k,\omega)$. The self-constant single loop $\sum(k,\omega)$ is given by (Wyld 1961)

$$\sum(k,\omega) = \frac{k^2}{(2\pi)^4}\int \mathrm{d}^3p\,\mathrm{d}\omega'\, b(\hat{k},p)C(p,\omega')G(k-p,\omega-\omega') \tag{6}$$

where $b(\hat{k}, \hat{p})$ is an angle-dependent dimensionless factor. The diagrammatic expansion for the correlation function can be written down but it yields no extra information beyond that contained in (6). The other independent entity is the expression for energy flux $\pi(k)$ from momentum k to $k + \mathrm{d}k$ and following Leslie, this is

$$\prod(k) = S\int_k^\infty \mathrm{d}k'k \int_{p+q=k'} \frac{\mathrm{d}^3 p}{\sum(p)+\sum(q)+\sum(k')} \times [b(\tilde{k}, \hat{p})\{C(p)C(q) - C(k)C(q)\} + p \leftrightarrow q]. \tag{7}$$

Between (6) and (7), we would have found the values of m and n of (3), but the infrared divergence of Edwards intervenes. Returning to (6) and considering the zero frequency shelf energy $\sum(k)$, we explore the contribution to the integral coming from the region $p \to 0$. The important range of the frequency integral is the low frequency region where $\omega \ll \sum(k)$ and consequently (6) yields, for the anticipated Kolmogorov spectrum,

$$\begin{aligned}\sum^2(k) &\simeq \frac{k^2}{(2\pi)^4}\int \mathrm{d}^3p\,\mathrm{d}\omega C(p,\omega) = \frac{k^2}{(2\pi)^3}\int \mathrm{d}^3 pC(p)\\ &\simeq K_0\frac{k^2}{(2\pi)^3}\int \mathrm{d}^3 pp^{-11/3}\\ &= K_0k^2k_0^{-2/3}\end{aligned} \tag{8}$$

where k_0 is a low momentum cut-off in the integral which diverges as $p \to 0$. The self-energy $\sum(k)$ thus obtained is a frequency which is the sweeping frequency corresponding to the sweeping time $\tau_s \sim \tau_0 k^{-1}k_0^{2/3}$. This is the characteristic time of advection of small eddies by the large ones. The problem of the divergence has come about because in (6) the wave numbers p and $k - p$ have coupled with extra strength for $p \to 0$ and consequently the dynamics has been dominated by the sweeping time.

We now note that (6) implies $\sum(k, \omega)$ vanishes for high frequency with the characteristic dynamic scaling behaviour (Ferrell *et al* 1968, Hohenberg and Halperin 1968)

$$\sum(k,\omega) \sim k^2(-\mathrm{i}\omega)^{(n-2)/n} \qquad \text{if } \omega \gg \sum(k). \tag{9}$$

The eddy viscosity is thus frequency dependent (Bhattacharjee and Scalapino 1981) giving rise to the known viscoelastic effect (Crow 1968). The corresponding correlation function is

$$C(k,\omega) = \frac{C(k)}{\sum(k)}\left[\frac{1}{i - \mathrm{i}\omega/\sum(k,w)} + \frac{1}{i + \mathrm{i}\omega/\sum^*(k,w)}\right] \tag{10}$$

for frequencies ω such that $\omega\tau_s \gg 1$. For lower frequencies, the viscoelastic effect is absent. We now return to (6) and with $C(k) \sim k^{-m}$ (3*a*) find, on performing the frequency integration,

$$\sum(k) = \frac{k^2}{(2\pi)^3}\int \mathrm{d}^2 pb(\hat{k},\hat{p})C(p)\left[1 + \frac{\sum(p)}{\sum^*(p,\omega) = -\mathrm{i}\sigma(k-p)}\right]^{-1}. \tag{11}$$

Comparing with the integrand of (8), we see that a screening factor, the square bracket, has been introduced. As $p \to 0$, we note that $\sum^+(p,\omega) = -\mathrm{i}\sum(k-p)$ becomes a high-frequency self-energy and must have the form $p^2[\sum(k)]^{1-(2/n)}$ and hence we have a screening factor of p^{2-n}. The convergence of the integral in (11) as $p \to 0$ is assured if $m + n < 5$. Under the conditions the scaling solution of (3) leads to (by a simple power counting)

$$m + 2n = 5. \tag{12}$$

Thus, the first of the Kolmogorov requirement has been satisfied. The dynamic coupling of the large eddies ($p \to 0$) and the small eddies ($p \sim 0(1)$) has been screened by the factor p^{2-n} coming from the viscoelastic effect. The second requirement of lossless cascade is met if the energy flux of (7) is independent of k and equal to the dissipation rate ϵ. This leads, by power counting to,

$$2m + n = 8 \tag{13}$$

leading to $m = 11/3$ and $n = 2/3$ as required. Thus, the screening approximation faithfully captures the physical picture of Kolmogorov and as expected leads to the Kolmogorov scaling.

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